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EXACT SOLUTIONS OF THE HAMILTON–JACOBI–BELLMAN EQUATION FOR PROBLEMS OF OPTIMAL CORRECTION WITH A CONSTRAINED OVERALL CONTROL RESOURCE†

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The problem of controlling the oscillations of a mathematical pendulum is considered. The overall control resource is subject to an integral constraint: the modulus of the control function to an arbitrary non-negative power (greater than or equal to unity) is a summable function over a specified time interval. The purpose of the control is to minimize a specified function of the phase variables to a fixed instant of time (Mayer's problem). Together with the deterministic case, a stochastic case is studied when the system is subject to random perturbations in the form of Gaussian white noise. In this case, it is required to minimize the mathematical expectation of specified functionals or to maximize the probability that a phase coordinate falls within a specified domain by a fixed instant of time. It is well known [1, 2] that the problem of constructing an optimal feedback control can be reduced to solving a Cauchy problem in an unbounded domain for the corresponding Hamilton–Jacobi–Bellman equation. It is proved that this problem is equivalent to a Cauchy problem for a linear parabolic equation. Exact solutions of this problem are found for the class of optimal control problems being considered. The case of a pulse correction, when the value of the integral of the modulus of the control function is bounded, is considered separately. The results obtained are extended to the case of an arbitrary number of phase variables if the control functions are square integrable.

1. FORMULATION OF THE PROBLEM

Suppose the controlled motion of a mass is described by the equations

$$\begin{aligned} &\dot{x}_1 = x_2, \quad \dot{x}_2 = -\omega^2 x_1 + u(t) + \sigma(t)\xi(t) \\ &x_1(0) = x_1^\circ, \quad x_2(0) = x_2^\circ \end{aligned} \tag{1.1}$$

Here t is the time, $0 \le t \le T, x_1$ and x_2 are phase coordinates, $u(t)$ is the control force (the control function), $\xi(t)$ is Gaussian white noise of unit intensity, $\sigma(t)$ is a bounded function which represents the intensity of a perturbation and ω is the natural frequency.

If $\sigma(t) = 0$, we shall call problem (1.1) a deterministic optimal control problem.

The following integral constraint is imposed on the magnitude of the control (the control function) $u(t)$

$$\int_0^T |u|^m dt \le Q_0^2, \quad Q_0 = \text{const} \tag{1.2}$$

Here m is a real positive number $m > 1, m = 2k/(2s-1), k \ge s, k, s = 1, 2, \dots$.

The case when $m = 1$ will be considered separately. The number m is the parameter of the problem: the different methods of specifying constraint (1.2) on the overall control resource correspond to

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different value of m. Note that the case when $m = 2$ is called control by a small traction, while the case when $m = 1$ is called pulse control.

We introduce the variable

$$q(t) = \int_t^T |u(t)|^m dt$$

The variable $q(t)$ has the meaning of the unconsumed control resource and $q(0) = Q_0^2$, $q(T) = 0$. Then, the equation

$$\dot{q} = -|u(t)|^m \tag{1.3}$$

can be added to Eqs (1.1).

The purpose of the control is to minimize one of the following functionals

$$E\{\varphi(x_1(T))\}, \quad E\{\varphi(x_2(T))\} \tag{1.4}$$

Here E is the sign of mathematical expectation and $\varphi(x)$ is a smooth, even, non-negative function of its arguments, and $\varphi'(x) > 0, x > 0$, $\varphi(0) = 0$. In the case of the deterministic problem ($\sigma = 0$), the sign of the mathematical expectation in functionals (1.4) must be discarded.

A typical example of functionals (1.4) is the potential and kinetic energy at the instant of time $t = T$, that is

$$\varphi(x_1) = \frac{1}{2}\omega^2 x_1^2, \quad \varphi(x_2) = \frac{1}{2}x_2^2$$

The problem of controlling system (1.1), (1.3) with the aim of maximizing the probability that the phase trajectory of the system will fall within a specified set N on the line x_1 or x_2 at the instant $t = T$ is a special case of the stochastic version of problem (1.4).

Next, we will assume that N is a connected set on the phase line x_1 and x_2 which is symmetrical about the origin of the coordinates.

The domains

$$x_1\colon |x_1| \le \delta_1, \quad x_2\colon |x_2| \le \delta_2, \quad \delta_1, \delta_2 = \text{const} > 0$$

serve as characteristics examples of the domains N.

By taking account of the specific features of functionals (1.4), the order of system (1.1), (1.3) can be reduced. In order to do this, we introduce the new variable

$$y(t) = x_2 \sin(\omega(T-t)) + \omega x_1 \cos(\omega(T-t))$$

It can be verified directly that

$$y(T) = \omega x_1(T), \quad \dot{y} = \sin(\omega(T-t))(\dot{x}_2 + \omega^2 x_1)$$

Consequently, in the case of functionals which depend solely on the final state of the phase variable x_1, system (1.1), (1.3) takes the form

$$\dot{y} = \sin(\omega(T-t))(u(t) + \sigma(t)\xi(t)), \quad \dot{q} = -|u|^m \tag{1.5}$$

If the variable

$$y(t) = x_2 \cos(\omega(T-t)) - \omega x_1 \sin(\omega(T-t))$$

is now introduced, then

$$y(T) = x_2(T)$$

and, hence, in the case of functionals which depend solely on the final state of the phase variable x_2, the first equation of system (1.5) takes the form

$$\dot{y} = \cos(\omega(T-t))(u(t)+\sigma(t)\xi(t))$$

If $\omega = 0$, then we consider $y(t) = x_2(T-t) + x_1$ as the new variable. Then, the first equation of system (1.5) takes the form

$$\dot{y} = (T-t)(u(t)+\sigma(t)\xi(t))$$

Generalizing the cases considered, we next consider the following equation of motion

$$\dot{y} = f(t)(u(t)+\sigma(t)\xi(t)), \quad \dot{q} = -|u|^m \tag{1.6}$$

where $f(t)$ is a smooth continuous function and $0 \le t \le T$.

Note that the problem has been solved numerically for the cases when $m = 1, 2$ for $\omega = 0$ using self-similar variables in [2, 3]. Local solutions of the corresponding Hamilton–Jacobi–Bellman equation have been investigated in [4].

2. THE HAMILTON–JACOBI–BELLMAN EQUATION ($m > 1$)

We will first consider the stochastic version of the initial problem. Suppose $S(y, q, t)$ is the minimum mathematical expectation of one of the functionals (1.4), which can be attained with the initial conditions $t = t_0, q = q_0, y = y_0$ in the optimal control problem described by the equations of state (1.6). Assuming that the function $S(y, q, t)$ exists and that it is sufficiently smooth, the Hamilton–Jacobi–Bellman (HJB) equation can be written as

$$S_t + \min_u\{f(t)uS_y - |u|^m S_q\} + \frac{1}{2}\sigma^2(t)S_{yy} = 0 \tag{2.1}$$

Here, the minimum is taken with respect to u. The function S satisfies the condition $S(y, q, T) = \varphi(y)$.

It follows from the formulation of the problem that the value of the function $S(y, q, T)$ can only decrease when the value of q is increased, since the greater the control resource the smaller the value which the functional can attain when the remaining conditions are the same, that is

$$S(y, q_2, t) = S(y, q_1, t), \quad q_1 < q_2$$

Taking into account the smoothness of the function $S(y, q, T)$, we obtain that the condition

$$S_q(y, q, T) < 0 \tag{2.2}$$

must be satisfied.

In the domain where $S_q(y, q, T) < 0$, a motion can be obtained using a control force and, in this case, a certain control resource q' is consumed. The minimum value of the expression in the braces in Eq. (2.1) is attained for the following control function

$$u = \left(\frac{|S_y f(t)|}{-mS_q}\right)^{\mu} \operatorname{sign}(S_y f(t)), \quad \mu = (m-1)^{-1} \tag{2.3}$$

After replacing the variable

$$\tau = \int_t^T f^2(s)ds \tag{2.4}$$

Eq. (2.1) becomes

$$S_\tau = \frac{1}{2}\sigma_1^2(\tau)S_{yy} + (m-1)g_m(\tau)\left(\frac{|S_y|}{-mS_q}\right)^{\mu+1} S_q \tag{2.5}$$

with the initial condition

$$S(y, q, 0) = \varphi(y) \tag{2.6}$$

Here

$$g_m(\tau) = |f(t)|^{\mu-1}, \quad \sigma_1(\tau) = \sigma(t)|f(t)|_{t=t(\tau)} \tag{2.7}$$

and the variables t and τ are related by (2.4). If $S_q(y, q, \tau) = 0$ for a certain set, then the minimum of the expression in the braces in Eq. (2.1) is attained when and only when either $u = 0$ or $S_q = S_y = 0$ simultaneously for this set. In the first case, uncontrolled motion under the action of random forces occurs. In these domains, Eq. (2.1) has the form

$$S_t + \frac{1}{2}\sigma^2(t)S_{yy} = 0 \tag{2.8}$$

In the second case, the control $u = 0$ and the linear-fractional function in Eq. (2.1), which contains the quantities S_y and S_q to the corresponding powers, must be determined in a set where $S_y = S_q = 0$. Since, according to our assumption, $\varphi(y)$ is an even function, problem (2.5), (2.6) is invariant under the replacement of the variable y by $-y$. Consequently, this problem can only be considered when $y > 0$ with the additional boundary condition

$$S_y(0, q, \tau) = 0 \tag{2.9}$$

All the arguments which have been presented also retain their meaning in the case of the problem of maximizing the likelihood that N falls within a specified set on the line x_1 or x_2 at the instant of time $t = T$. Calculation of the minimum in Eq. (2.1) has to be replaced by calculation of the maximum. Note that a maximum of the above-mentioned expression will exist when and only when $S_q(y, q, \tau) > 0$.

The form of formula (2.3) is preserved in the deterministic case.

Equation (2.5) will take the form

$$S_\tau = (m-1)p_m(\tau)\left(\frac{|S_y|}{-mS_q}\right)^{\mu+1} S_q \tag{2.10}$$

Here,

$$\tau = T - t, \quad p_m(\tau) = |f(T-\tau)|^{\mu-1} \tag{2.11}$$

3. EXACT SOLUTIONS OF THE HAMILTON–JACOBI–BELLMAN EQUATION ($m > 1$)

Consider the case of the deterministic system $\sigma(t) = 0$.

Assertion 1. The exact solution of Eq. (2.1) is given by the equality

$$S(y, q, \tau) = \varphi(z)$$

where

$$z = y - P_m(q, \tau), \quad P_m(q, \tau) = q^{1/m}(\Theta_m(\tau))^{1-1/m}, \quad \Theta_m(\tau) = \int_0^\tau p_m(s)ds$$

The function $p_m(s)$ is defined by equality (2.11). The optimal feedback control in the deterministic optimal control problem, which is defined by the equations of state (1.6) with one of the functionals (1.4), is determined using the formula

$$u = \begin{cases} -|f(t)|^{\mu}\left(\dfrac{q}{\Theta_m(\tau)}\right)^{1/m} \operatorname{sign} f(t), & y \ge P_m(q, \tau) \\ 0, \quad 0 \le y < P_m(q, \tau) \end{cases} \tag{3.1}$$

Proof. The first part of the assertion can be verified directly by substituting the function $\varphi(z)$ into Eq. (2.10).

We consider the domains

$$D_1^m = \{y, q, \tau: y > P_m(q, \tau)\}, \quad D_2^m = \{y, q, \tau: 0 \le y < P_m(q, \tau)\} \tag{3.2}$$

The boundary γ_m of these domains is specified by the surface $y = P_m(q, \tau)$, which contains the coordinate axis $q = 0$, and its sections when $q = \text{const} > 0$ are a monotonically increasing curve emerging from the origin of coordinates in the (y, τ) plane. The surface divides the domains D_1^m and D_2^m in such a way that the surface $\tau = 0$ is the boundary of the domain D_2^m, and the surface $y = 0$ is the boundary of the domain D_1^m.

Equation (2.1) and condition (2.6) satisfy the boundary condition

$$S_q(y, q, \tau) < 0, \quad u \ne 0$$

Consequently, the first part of formula (3.1) only holds in the domain D_1^m. The function $S = \varphi(z)$, together with its derivatives with respect to y and q, vanish on the boundary γ_m of the domains D_1^m and D_2^m.

We extend the function $S = \varphi(z)$ with a zero into the domain D_2^m. The choice of the control $u = 0$ in the domain D_2^m ensures that the phase trajectory of the system falls within the set $y = 0$ when $\tau = 0$.

In fact, it follows from Eqs (1.6) when $u = 0$ that $y = \text{const} > 0$, $q = \text{const} > 0$ and, hence, when the inverse time, $\tau = T - t$, is decreased, the phase trajectory of the system necessarily falls on the boundary γ_m (see Fig. 1).

The trajectories of the deterministic problem (1.6) in the domain D_1^m lie on the surface

$$G(y, q, \tau) = y - P_m(q, \tau) = \text{const}$$

In order to prove this, it is necessary to consider the normal to this surface and to check that it is orthogonal to the vector composed of the right-hand sides of Eqs (1.6). Taking formula (3.1) into account, we write the deterministic system (1.6) in the form

$$\dot{y} = -F, \quad \dot{q} = -F^m, \quad \dot{\tau} = -1, \quad F = (f(\tau))^{\mu}(q/\Theta_m(\tau))^{1/m}$$

Using the explicit form of the expression for the function $\Theta_m(\tau)$, it can be verified that the scalar product of the vectors ∇G and $(-F, -F^m, -1)$ is equal to zero.

Formula (3.1) also retains its meaning on the surface γ_m itself, despite the fact that $S_q = S_y = 0$ on γ_m, since the indeterminacy in expression (3.1) which arises here can be expanded. Hence, the possibility mentioned earlier in Section 2 for which $S_y = S_q = 0$ and the control $u \ne 0$ is realized on the set γ_m.

To sum up, in the domain D_1^m together with the boundary γ_m, motion occurs over the surface $y - P_m(q, \tau) = \text{const}$ and, in the domain D_2^m, the phase trajectory has the form $y = q = \text{const}$ which ensures that it falls within the set $y = 0$ by the instant $\tau = 0$ (see Fig. 1).

We will now consider the stochastic case of problems (1.6).

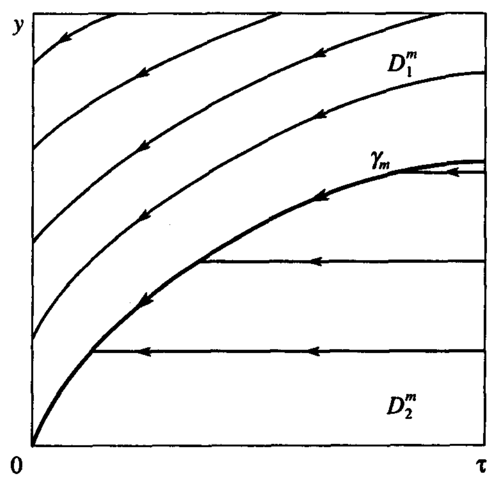

Fig. 1

Assertion 2. The Hamilton–Jacobi–Bellman equation (2.5) reduces to the linear parabolic equation

$$\Phi_\tau = \frac{1}{2}\sigma_1^2(\tau)\Phi_{ww} \tag{3.3}$$

in the function Φ of the two variables w and τ, where

$$w = y - R_m(q, \tau), \quad R_m(q, \tau) = q^{1/m}(\psi_m(\tau))^{(m-1)/m}, \quad \psi_m(\tau) = \int_0^\tau g_m(s)ds \tag{3.4}$$

(the function $g_m(s)$ is given by equality (2.7) for τ and t which satisfy equality (2.4)). The optimal feedback control in the stochastic problems (2.5), (2.6) is determined using the formula

$$u = \begin{cases} -|f(t)|^{1/(m-1)}\left(\dfrac{q}{\psi_m(\tau)}\right)^{1/m} \operatorname{sign} f(t), & y \ge R_m(q, \tau) \\ 0, \quad 0 \le y < R_m(q, \tau) \end{cases} \tag{3.5}$$

Proof. The first part of the assertion can be verified by direct substitution. The solution of the Cauchy problem (2.5), (2.6) has the form

$$S = \Phi(w, \tau) = \int_0^\infty A(w, \tau, \eta)\varphi(\eta)d\eta \tag{3.6}$$

where

$$\begin{aligned} A(w, \tau, \eta) &= \frac{1}{2\sqrt{\pi B(\tau)}}\left[\exp\left(-\frac{(w-\eta)^2}{2B(\tau)}\right) + \exp\left(-\frac{(w+\eta)^2}{2B(\tau)}\right)\right] \\ B(\tau) &= \int_0^\infty \sigma_1^2(s)ds \end{aligned} \tag{3.7}$$

We consider the domains Ω_1^m and Ω_2^m, which differ from the corresponding domains (3.2) in that the function $P_m(q, \tau)$ is replaced by $R_m(q, \tau)$.

As in the deterministic case, it is proved that the surface $\Gamma_m = \{y, q, \tau: y - R_m(q, \tau) = 0\}$ divides the domains Ω_1^m and Ω_2^m. The synthesis of the optimal control in the domain Ω_1^m, which includes the boundary Γ_m, is determined using the upper part of formula (3.5). We put $u = 0$ in the domain Ω_2^m. A point mass then executes uncontrolled motion, which represents a random walk in the set

$$0 \le y < R_m(q, \tau)$$

Since $R_m(q, \tau) \to 0$ when $\tau \to 0$, the phase coordinate of the point, with a probability of unity at a certain instant of time $\tau > 0$, falls onto the boundary Γ_m where correction occurs. In this case, the value of the Bellman function $S(y, q, \tau)$ in the domain Ω_2^m is determined from the solution of the following boundary-value problem

$$\begin{aligned} &S_\tau = \frac{1}{2}\sigma_1^2(\tau)S_{yy}, \quad y > 0, \quad q > 0, \quad \tau > 0 \\ &y = R_m(q, \tau): S = \Phi(0, \tau); \quad y = 0, \quad S_y|_{y=0} = 0 \end{aligned} \tag{3.8}$$

where $\Phi(w, \tau)$ is the solution (3.6) of Eq. (3.3).

The result of Assertion 2 is also preserved in the case of the problem of maximizing the incidence of N on a phase line in a specified domain. We supplement Eq. (2.5) with the initial conditions

$$S(y, q, 0) = \begin{cases} 1, & y \in N \\ 0, & y \notin N \end{cases}$$

The solution of the resulting Cauchy problem has the form

$$S = \int_N A(w, \tau, \eta)\varphi(\eta)d\eta$$

Here w is a variable which is defined by the first equality of (3.4) and $A(w, \tau, \eta)$ is the function given be expression (3.7). Integration is carried out over the symmetric set N_m, where $y \geq R_m(q, \tau)$. All the remaining arguments are a repetition of the arguments used in the proof of Assertion 2. Note that a local solution of Eq. (2.5) in the domain Ω_1^m, $m = 2$ was previously found in [4].

Examples. 1. We will consider the stochastic problem with $f(t) = T - t$. It follows from expression (2.4) that $\tau = (T-t)^3/3$. The solution of Eq. (2.5) in accordance with Assertion 2 is a function of the variables

$$w = y - R_m(q, \tau)$$

$$R_m(q, \tau) = ((m-1)/(2m-1))^{1-1/m}(3\tau)^{2/3-1/(3m)}q^{1/m}$$

The optimal feedback control in this case is determined using the formula

$$u = \begin{cases} -(q(3\tau)^{-1/3})(2m-1)/(m-1))^{1/m}, & y \geq R_m(q, \tau) \\ 0, \quad 0 \leq y < R_m(q, \tau) \end{cases}$$

2. We will consider another common case. We put $m = 2$ and $f = \sin\omega(T-t)$; it then follows from equality (2.4) that the variables t and τ are connected by the relation

$$\tau = \frac{1}{2}(T - t - (\sin 2\omega(T-t))/(2\omega)) \tag{3.9}$$

In this case, the optimal control is given by

$$u = \begin{cases} -\sin(\omega(T-t))(q/\tau)^{1/2}, & y \geq \sqrt{q\tau} \\ 0, \quad 0 \leq y < \sqrt{q\tau} \end{cases}$$

3. We will now present an example of a deterministic situation. For the function $f = T - t = \tau$, from relation (2.11) in accordance with Assertion 1 we obtain

$$p_m(\tau) = \tau^{\mu+1}, \quad \Theta_m(\tau) = (2+\mu)^{-1}\tau^{2+\mu}$$

Formula (3.1) (the optimal feedback control) takes the specific form

$$u = \begin{cases} -\tau^{-1/m}(2+\mu), & y \geq q^{1/m}(\Theta_m(\tau))^{1-1/m} \\ 0, \quad 0 \leq y < q^{1/m}(\Theta_m(\tau))^{1-1/m} \end{cases}$$

4. We put $m = 2$. Then, for the function $f = \sin\omega(T-t) = \sin\omega\tau$, the corresponding control has the form (3.1), where

$$\Theta_m(\tau) = (\tau + \sin(2\omega\tau)/(2\omega))/2$$

$$p_m(\tau) = ((\tau + \sin(2\omega\tau)/(2\omega))/(2q))^{1/2}$$

A graph of the change in the optimal control u and the optimal phase trajectory of a deterministic system (see the example) in the case when $m = 2$, $T = 10$ and $f = T - t$ are shown in Fig. 2. The initial data for system (1.6) in this example have the form

$$x_1^0 = 4, \quad x_2^0 = 2, \quad q_0 = 1$$

The "plus" sign is taken in relation (1.3).

The same characteristics are shown in Fig. 3 for $m = 2$, $T = 10$, $f = \cos(\pi(T-t)/3)$ for system (1.6) with the initial data $x_1^0 = 2, x_2^0 = 8, q_0 = 1$.

The calculations show that the optimal phase trajectories of a system with a non-zero value of ω have the form of unwinding spirals. Hence, there is an increase in the kinetic energy in the problem of minimizing the potential energy of a system to a specified instant of time, while there is an increase in the potential energy (Fig. 3) in the analogous problem of minimizing the kinetic energy to the final instant of time.

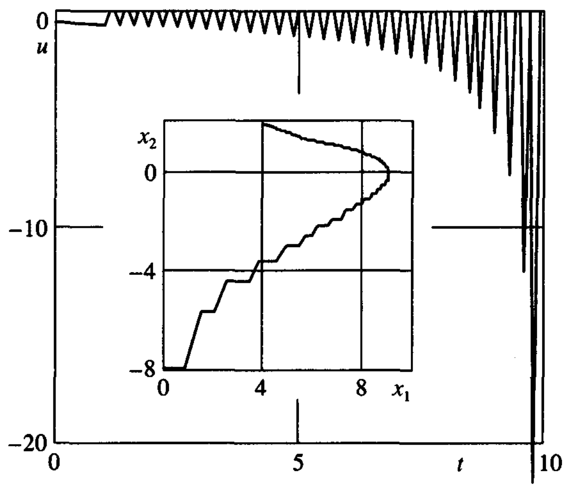

Fig. 2

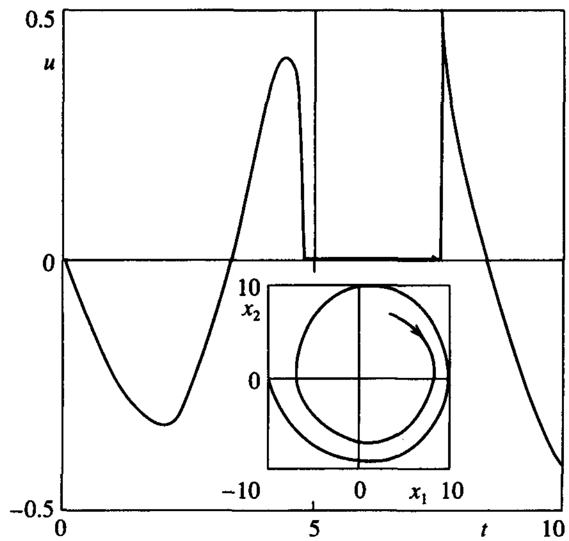

Fig. 3

4. THE LIMITING CASE - PULSE CORRECTION

It is clear from the form of the Hamilton–Jacobi–Bellman equation (2.5) that the case when $m = 1$ is special. It has been shown in [1] that the form of Eq. (2.5) on taking the limit as $m \to 1$ depends on the value of the following expression

$$H(y, q, \tau) = f(t)S_y + S_q$$

and $H(y, q, \tau) \le 0$ for all permissible values of y, q and τ.

In the domain Ω_2^1, where the inequality $H(y, q, \tau) < 0$ is satisfied, there is uncontrolled motion under the action of random forces. The function S in this situation is the solution of the Cauchy problem for the limiting parabolic equation

$$S_\tau - \frac{1}{2}\sigma_1^2(\tau)S_{yy} = 0$$

The equality $H(y, q, \tau) = 0$ is satisfied in the domain Ω_1^1. In this domain, the function $S(y, q, \tau)$ is the solution of the first-order hyperbolic equation

$$f(\tau)S_y + S_q = 0 \tag{4.1}$$

The characteristics of this equation are specified by the equality $y - f(\tau)q$ = const. The latter means that there is a pulse correction in the domain Ω_1^1 under the action of which a point is instantaneously displaced along the characteristics of hyperbolic equation (4.1). At the same time, as a result of the correction, the point either finds itself on the boundary Γ_1 of the domains Ω_1^1 and Ω_2^1 or the control resource is completely exhausted. The determination of the boundary completely solves the problem of constructing the optimal feedback control.

The problem was studied earlier in [1–3] for the case when $\omega = 0$ and, in [4–5], solutions were found numerically using self-similar variables. An analogous problem was investigated in [6, 7] using variational inequalities. A game approach to the problem for the deterministic case was considered in [8].

Using the results of Section 3, we will now study the limiting position of the boundary when $m \to 1$. Consider the continuous function

$$F(\tau) = \max|f(s)|, \quad 0 \le s \le \tau \tag{4.2}$$

Here, the variables t and τ are related by (2.4).

Assertion 3. Suppose $f(\tau)$ is a non-negative, monotonically increasing function of the variable τ, $0 \le \tau \le T$. Then, when $m = 1$, the solution of Eq. (2.5) in the domain Ω_1^1 is a function of the two variables: $S = \Phi(w, \tau)$, where $w = y - qF(\tau)$. The function $\Phi(w, \tau)$ is the solution of the linear parabolic equation (3.3): it is defined by formula (3.6), (3.7). In the domain Ω_1^1, there is a pulse correction under the action of which a phase point is displaced over the surfaces $y - qF(\tau)$ = const. In the domain $\Omega_2^1 = \{y, q, \tau\colon 0 \le y < qF(\tau)\}$, uncontrolled motion under the action of random forces occurs. The boundary of the domains Ω_1^1 and Ω_2^1 is determined by the equality $y = qF(\tau)$, and the Bellman function $S(q, y, \tau)$ in the domain Ω_2^1 is the solution of the boundary-value problem (3.8), where $R_{q,\tau} = qF(\tau)$.

Proof. We will now make use of the result of Assertion 2 while letting m tend to unity. To do this, it is necessary to calculate the limit of the expression $R_m(q, \tau)$, which is defined by the second and third equalities of (3.4). Using relations (2.4) and (2.7), we obtain

$$F(\tau) = \lim_{m \to 1}\left(\int_0^\tau ((f(s))^{2/m-1})^{m\mu}\right)^{(m-1)/m}$$

It is well known [9] that the equality

$$\lim_{p \to \infty}\left(\int_a^b |\varphi(s)|^p\right)^{1/p} = \max_{a \le s \le b}|\varphi(s)| \tag{4.3}$$

holds for any continuous function $\varphi(s)$.

Taking account of (4.2) and (4.3) we obtain $F(\tau) = f(\tau)$. Taking the limit in the second and third formulae of (3.4) we conclude that the $S(q, y, \tau)$-function in the domain Ω_1^1 satisfies Eqs (3.8).

Remark. If the condition that the function $f(\tau)$ is monotonic is dropped, then $f(\tau) \le F(\tau)$, and equality will only be achieved at points where the function $f(\tau)$ reaches a maximum. In this case, pulse correction does, in fact, happen at these instants of time.

Examples. 1. Suppose $\omega = 0$ and $f = T - t$. It follows from expression (2.4) that the variables t and τ are connected by the relation $\tau = (T - t)^3/3$. Consequently, $F(\tau) = (3\tau)^{1/3}$.

2. If $\omega \ne 0$ and $f = \sin(\omega(T - t))$, it follows from expression (2.4) that the connection between the variables t and τ is achieved using equality (3.9). In this case,

$$F(\tau) = \begin{cases} \sin(\omega\tau), & 0 \le \tau \le \pi/(2\omega) \\ 1, & \tau > \pi/(2\omega) \end{cases}$$

The domain Ω_1^1 consists of the set $y \ge \sin(\omega\tau)$ if $0 \le \tau \le \pi/(2\omega)$. If, however, $\tau > \pi/(2\omega)$, then a pulse correction is put into effect at the discrete instants of time $\tau_k = k\pi/(2\omega)$ ($k = 1, 2, \ldots$). Motion occurs in a direction opposite to the direction of the y axis. In this case, either the phase point falls in the set $y = q\sin(\omega\tau)$ or the control resource is completely used up.

A graph of the change in the control resource and the optimal phase trajectory are shown in Fig. 4 for the case when $m = 40/39$ and $T = 8$ in a problem with $f(t) = \sin(\pi(T - t))/3$, that is, a situation when the value of m is close to unity. The initial data of system (1.6) are: $x_1^0 = 8$, $x_2^0 = 4$, $q_0 = 0.8$. The optimal control (Fig. 4) is a train of pulses. As in the preceding cases, the phase trajectory of the system has the form of an unwinding spiral.

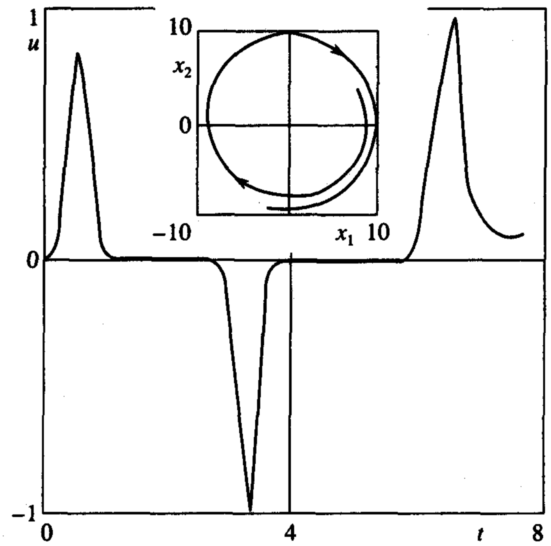

Fig. 4

5. THE CASE OF AN ARBITRARY NUMBER OF DEGREES OF FREEDOM ($m = 2$)

The fact that the Hamilton–Jacobi–Bellman equation is integrable (Assertion 2) enables us to solve certain multidimensional problems.

Consider a system with n degrees of freedom, described by the equations

$$\begin{aligned}
&\dot{X}_1 = X_2, \quad \dot{X}_2 = -KX_1 + U(t) + \sigma(t)\xi(t) \\
&X_k(t) = (x_1^k(t), x_2^k(t), \dots, x_n^k(t)), \quad k = 1, 2, \\
&U(t) = (u_1(t), u_2(t), \dots, u_n(t))
\end{aligned} \tag{5.1}$$

Here $X_1(t)$ and $X_2(t)$ are n-dimensional displacement and velocity vectors, $U(t)$ is a control vector and its components are the control forces, K is a symmetric positive-definite matrix, $\xi(t)$ is the vector of independent Gaussian white noises of unit intensity and $\sigma(t)$ is a function which specifies the intensity of the perturbation by the random forces. The constraint

$$\sum \int_0^T u_i^2 dt \le Q_0^2, \quad Q_0 = \text{const} \tag{5.2}$$

is imposed on the control forces.

Here and everywhere henceforth, summation is carried out from $i = 1$ to $i = n$.

The problem arises of finding the control for which either the mathematical expectation of the potential energy of the system

$$J_1 = \frac{1}{2}(KX_1, X_1)$$

or the kinetic energy

$$J_2 = \frac{1}{2}(KX_2, X_2)$$

reach the minimum value in a fixed instant of time $t = T$.

A non-degenerate orthogonal transformation exists [10] such that

$$Z_1 = A^T X_1, \quad Z_2 = A^T X_2, \quad A^{-1} = A^T, \quad A^T K A = \Lambda$$

Here Λ is a diagonal matrix with diagonal elements $\lambda_i = \omega_i^2$, where ω_i are the frequencies of free vibrations of the initial system. System (5.2) takes the form

$$\begin{aligned} \dot{Z}_1 &= Z_2, \quad \dot{Z}_2 = -\Lambda Z_1 + V(t) + \sigma(t)A^T\xi(t) \\ Z_k(t) &= (z_1^k(t), z_2^k(t), \ldots, z_n^k(t)), \quad k = 1, 2 \\ V(t) &= (\upsilon_1(t), \upsilon_2(t), \ldots, \upsilon_n(t)) \end{aligned} \tag{5.3}$$

Here, $V(t) = A^TU(t)$ and $V(t)$ is a new control.
The control υ_i also satisfies constraints (5.2).
In fact, we have

$$\sum\int_0^T \upsilon_i^2 dt = \int_0^T (V, V)dt = \int_0^T (AU, A^TU)dt = \int_0^T (U, U)dt \le Q_0^2$$

The functionals J_1 and J_2 retain their own kind of quadratic forms

$$J_1 = \frac{1}{2}(Z_1, Z_1), \quad J_2 = \frac{1}{2}(Z_2, Z_2)$$

We introduce the variable

$$q(t) = \int_0^T (\sum \upsilon_i^2)dt$$

Taking account of constraint (5.2), we obtain

$$\dot{q} = -\sum \upsilon_i^2, \quad q(0) \le Q_0^2, \quad q(T) = 0 \tag{5.4}$$

We now put

$$y_i(t) = z_i^2 \sin\omega_i(T-t) + \omega_i z_i^1 \cos\omega_i(T-t)$$

in the case of the functional J_1. In the case of the functional J_2, we introduce the variable

$$y_i(t) = z_i^2 \cos\omega_i(T-t) + \omega_i z_i^1 \sin\omega_i(T-t)$$

If, however, $\omega = 0$ and $f_i = T - t$, we consider

$$y_i(t) = z_i^2(T-t) + z_i^1$$

System (5.3) can be written in the form

$$\begin{aligned} \dot{Y}(t) &= f(t)(V(t) + \sigma(t)A\xi(t)) \\ Y(t) &= (y_1(t), y_2(t), \ldots, y_n(t)) \end{aligned} \tag{5.5}$$

The Hamilton–Jacobi–Bellman equation for problem (5.4), (5.5) has the form

$$S_t + \min_{\upsilon_1, \upsilon_2, \ldots, \upsilon_n} \{\sum (f_i(t)\upsilon_i S_{y_i} - \upsilon_i^2 S_q)\} + \frac{1}{2}\sigma^2(t)\sum f_i^2(t) S_{y_i y_i} = 0 \tag{5.6}$$

Here $S(y_1, y_2, \ldots, y_n, q, t)$ is the Bellman function. The condition

$$t = T: S = \frac{1}{2}\sum y_i^2$$

has to be added to this equation.
A minimum of the expression on the left-hand side of equality (5.6) only exists if $S_q \le 0$. On calculating this minimum, we obtain

$$\upsilon_i = f_i(t)\frac{S_{y_i}}{2S_q}$$

and are therefore write Eq. (5.6) in the form

$$S_t + \frac{1}{4}\sum f_i^2(t)\frac{S_{y_i}^2}{S_q} - \frac{1}{2}\sum \sigma^2(t) f_i^2(t) S_{y_i y_i} = 0$$

Since the case when $\omega = 0$ is being considered, after making the substitution $\tau = (T-t)^3/3$ the preceding equation takes the form

$$S_\tau = \frac{1}{4}|\nabla S|^2/S_q + \frac{1}{2}\sigma_1^2(\tau)\Delta S \tag{5.7}$$

Here Δ is the n-dimensional Laplace operator and $\nabla S = (S_{y_1}, S_{y_2}, \dots, S_{y_n})$.

We convert the equation when $t = T$ into the following condition when $\tau = 0$

$$\tau = 0: S = \frac{1}{2}\sum y_i^2 \tag{5.8}$$

and we introduce the variable

$$r = \left(\sum y_i^2\right)^{1/2}$$

The solution of Eq. (5.7) with the initial condition (5.8) is a function of the three variables r, q and τ. Actually, on putting $S = \Phi(r, q, \tau)$, we obtain

$$\Phi_\tau = \frac{1}{4}\frac{\Phi_r^2}{\Phi_q} + \frac{1}{2}\sigma_1^2(\tau)(\Phi_{rr} + (n-1)r^{-1}\Phi_r)$$

from relations (5.7) and (5.8).

It follows from Assertion 2 that $\Phi(r, q, \tau)$ is a function of the two variables: τ and $w = r - \sqrt{q\tau}$, that is, $\Phi(r, q, \tau) = F(w, \tau)$, and the function $F(w, \tau)$ is the solution of the parabolic equation

$$F_\tau = \frac{1}{2}\sigma_1^2(\tau)(F_{ww} + (n-1)r^{-1}F_w)$$

The optimal feedback control in the above problem is determined using the formula

$$\upsilon_i = \begin{cases} -\sqrt{q\tau^{1/3}}, & r \geq \sqrt{q\tau} \\ 0, & 0 \leq r < \sqrt{q\tau} \end{cases}$$

Hence, the components of the control vector $V(t)$ (5.3) are the control functions υ_i, and the initial control functions $u_1, u_2, \dots, u_k$ are uniquely recovered using the orthogonal transformation A.

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